



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

whose degree is obtained by dividing the order of G by the order of this subgroup. When the subgroup is a maximum subgroup the corresponding transitive group is primitive; when this condition is not satisfied it is imprimitive*. All subgroups of this type that can be made to correspond in a simple isomorphism of G to itself, and only these, lead to the same transitive group. Hence we can readily obtain all the transitive substitution groups that are determined by G , i. e. those which are simply isomorphic to G .

From what precedes it follows that the complete study of a group (G) implies the study of its successive groups of isomorphisms and those of cogredient isomorphisms, its selfconjugate subgroups and the corresponding quotient groups, its successive derivatives and the corresponding solvable groups, its characteristic and cogredient subgroups, its subgroups that do not involve any selfconjugate subgroup besides identity and the corresponding transitive substitution groups, etc. From this standpoint each group generally determines a group complex whose various parts throw much light on the structure of the group.

Chicago, August, 1898.

*Dyck, *Mathematische Annalen*, vol. 22, 1883, page 90.

INFINITY, THE INFINITESIMAL, AND ZERO.

By HENRY HEATON, M. Sc., Atlantic, Iowa.

So much has been written upon this subject that I do not flatter myself that I can write anything new. I shall only attempt to point out a few of the things that have been written that to my mind plainly cannot be true, and with these errors in view endeavor to express the truth as I see it. If in so doing I fall into error I shall have the satisfaction of knowing that greater men have done the same.

In the *Analyst*, Vol. VIII., pages 105-113, Professor Judson has a very interesting article upon this subject in which he quotes freely from many distinguished authors many things that are plainly fallacious. Yet when he attempts to outline his view of the subject it seems to the writer that he blunders fully as badly as those whom he criticises.

He says: "(7) *A variable which decreases indefinitely and which by reason of its indefiniteness, may be considered as less than any assignable value, is called an infinitesimal.* We shall make use of a horizontal 0 to represent an infinitesimal. Thus, read when $x=0$, when x decreases indefinitely or when x is an infinitesimal.

(8) If a be a constant the expression $a/0=\infty$ and $a/\infty=0$ are rigidly exact."

He says further, " $\infty \pm a = \infty$, also $0 \times \infty = 0/0$ is wholly indeterminate. We cannot write $a \pm 0 = a$, nor can we write $a \pm 0 = c$, if c is a constant."

He quotes Professor DeMorgan with evident approval as saying that he dates his first clear conception of mathematical infinity from the time when he rejected the relation $a/0 = \infty$.

He says: 'Instead of saying the 'tangent is infinite when $x = 90^\circ$ ' we may say 'the tangent of x becomes infinite as x approaches 90° '. In tracing curves, if $y = x^3/(a-x)$, we should say, y becomes infinite as x approaches a , and not when $x = a$.'

Dr. Davies, in his Differential Calculus, Art. 56. says: "When therefore each part becomes *infinitely small*, any *finite* number of them is 0, but an infinite number of them is equal to a finite quantity."

If a *finite* number of them equals 0, how about *one* of them?

In Lilley's Higher Algebra under the interpretation of $a/0$ I find the following remarkable statement. "Dividing 12 by a number that decreases by 1 each time commencing with +4, we have $\frac{12}{+4} = 3$, $\frac{12}{+3} = 4$, $\frac{12}{+2} = 6$, $\frac{12}{+1} = 12$

$\frac{12}{0} = 0$, etc. 3 means that 3 times +4 can be subtracted from 12 and leave 0; 4 means that 4 times +3 can be subtracted from 12 and leave 0; the quotient 0 means that there is *no number of times zero* that the divisor, 0, can be subtracted from 12 and *leave zero*." His general conclusion is then that $a/0 = 0$.

He says: "If a constant be divided by an infinitesimal, the quotient is infinity, and if a constant be divided by infinity, the quotient is infinitesimal." Thus agreeing with Judson.

His conclusion that $a/0 = 0$ is the result of a mere play upon words. He overlooks the fact that if $a/0 = 0$ then if 0 be subtracted from a 0 times the result must be 0.

I am willing to agree with these writers that $a/0$ is the symbol of impossibility; but I would say that ∞ is also the symbol of impossibility. Hence, $a/0 = \infty$.

a/x does not equal infinity so long as x has the slightest shadow of value.

I believe that mathematical infinity is something beyond the ken of even a Professor DeMorgan, that he was never so far from having a correct conception of it as when he claimed to have a clear conception of it. $\tan x$ does not become infinite as x approaches 90° , for so long as x differs by the shadow of a hair from 90° , $\tan x$ is finite.

I think I know exactly what is meant by the term zero. But I can have no conception either of infinity or of the infinitesimal, and I think it would be well if mathematicians would let both pretty severely alone.

If $u = x^2$ and $du =$ an increment that u receives as a result of an increment dx to the value of x ; then $du = 2x dx + dx^2$, and $\frac{du}{dx} = 2x + dx$. This equation cannot reduce to $\frac{du}{dx} = 2x$ in any other way than by putting dx , and consequently

du =absolute zero. If dx has any value at all, infinitesimal or otherwise, $\frac{du}{dx}$ does not equal 2x.

If $u=x^n$, the general expression for $\frac{du}{dx}$, supposing du and dx to have definite values, is $\frac{du}{dx}=nx^{(n-1)} + \frac{n(n-1)}{1.2}x^{(n-2)}dx + \frac{n(n-1)(n-2)}{1.2.3}x^{(n-3)}dx^2$, etc. When $dx=0$, and not before, this reduces to $\frac{du}{dx}=nx^{(n-1)}$. This is called the differential coefficient of u with respect to x .

It may be ridiculous to say 0/0 equals anything. But there can certainly be no objection to saying that the differential coefficient of a function u with respect to x , is the value to which the *general expression for $\frac{du}{dx}$* reduces when dx is equated to 0, for this is what the differential coefficient really is.

It may be readily shown that the differential coefficients, or derivatives, of two equal functions are always equal, and that if two derivatives are equal *their primitives can differ only by a constant quantity*.

The results obtained by integration are correct because of this last truth.

The process called integration, instead of being the summing of an infinite number of infinitely small quantities, is really the process of finding the primitive corresponding to a given derivative.

As an illustration I shall compute the variable volume, V , of a right cone whose variable altitude is x and the radius of whose variable base is $\frac{ax}{h}$. If dV is any increment which V receives as a result of an increment dx to x , then $dV=\frac{a^2x^2\pi}{h^2}dx+p(dx^2)$. Where p is a variable finite quantity known to be less than the circumference whose radius is $\frac{ax}{h}$. Hence $\frac{dV}{dx}=\frac{a^2x^2\pi}{h^2}+pdx$.

When dx is equated to zero the expression for $\frac{dV}{dx}$ reduces to $\frac{a^2x^2\pi}{h^2}$. Hence, $\frac{a^2x^2\pi}{h^2}$ is the derivative of V with respect to x . But $\frac{a^2x^2\pi}{h^2}$ is the derivative of $\frac{a^2x^3\pi}{3h^2}+C$. Hence $V_0r=\frac{a^2h\pi}{3}$.